A Ramsey type problem for highly connected subgraphs

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Overview

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   - Ramsey Theory
   - The Connectivity Version
2. Our Progress
   - The Decomposition
   - The Proof
   - The Counterexample
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Ramsey Theory

Ramsey Theory

Definition: Ramsey Number

For any given integers s, t, the Ramsey number $R(s, t)$ is the smallest integer n, such that for any 2-edge-colored (red/blue) K_n, there must exist a red K_s or a blue K_t.

Theorem (Ramsey, 1930)

For any given integers s, t, the Ramsey number $R(s, t)$ exists.

Ramsey Theory

Theorem

For any given integers $s_1, s_2, \ldots, s_c$, there exist Ramsey number $R(s_1, s_2, \ldots, s_c)$, such that for any c-edge-colored K_n where $n \geq R(s_1, s_2, \ldots, s_c)$, there must exist a K_{s_i} in color i.

The Connectivity Version

Definition: k-connected

A graph is k-connected if and only if it has more than k vertices and does not have a vertex cut of size at most $k-1$.

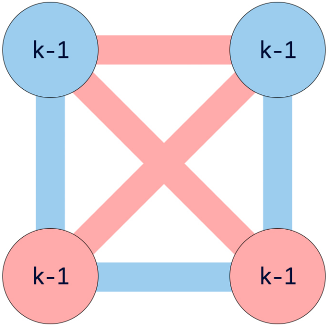
Connectivity version Ramsey number: $r_c(k)$

Let $r_c(k)$ denote the smallest integer such that every c-edge-colored complete graph on $r_c(k)$ vertices must contain a k-connected monochromatic subgraph.

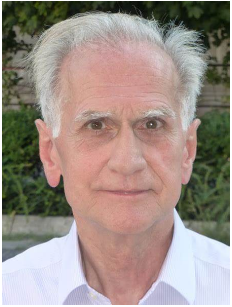
The Connectivity Version

Theorem (Matula, 1983)

- $2c(k-1)+1 \leq r_c(k) < (10/3)c(k-1)+1.$
- $4(k-1)+1 \leq r_2(k) < (3+\sqrt{11/3})(k-1)+1.$



The Conjecture by Bollobás and Gyárfás



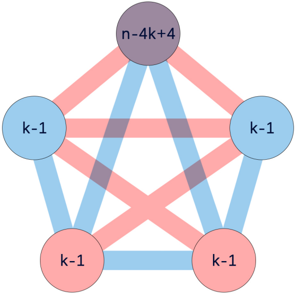
Béla Bollobás

András Gyárfás

The Conjecture by Bollobás and Gyárfás

Conjecture (Bollobás and Gyárfás, 2008)

Let k, n be positive integers. For $n > 4(k-1)$, every 2-edge-colored K_n contains a k-connected monochromatic subgraph with at least $n - 2k + 2$ vertices.



Known Results

Conjecture (Bollobás and Gyárfás, 2008)

Let k, n be positive integers. For $n > 4(k-1)$, every 2-edge-colored K_n contains a k-connected monochromatic subgraph with at least $n - 2k + 2$ vertices.

- True for $k \leq 2$; Sufficient to prove the conjecture holds for $4k - 3 \leq n < 7k - 5$. (Bollobás and Gyárfás, 2008)
- True for $k = 3$ and $n \geq 13k - 15$. (Liu, Morris, and Prince, 2009)
- True for $n > 6.5(k-1)$ (Fujita and Magnant, 2011)
- True for $n > 4(k-1)$??? (Łuczak, 2016)

Our Progress

Theorem (Lo, Wu & Xie, 2023+)

- For every $k \in \mathbb{Z}^+$, let $n = \lfloor 5k - \sqrt{8k - \frac{31}{4}} - 2.5 \rfloor$. There exists a 2-edge-colored K_n, such that there is no k-connected monochromatic subgraph, which contains at least $n - 2k + 2$ vertices.
- Let $n, k \in \mathbb{Z}^+$, $k \geq 16$. If $n > 5k - \sqrt{8k - \frac{31}{4}} - 2.5$, then for any 2-edge-colored K_n, there exists a k-connected monochromatic subgraph, which contains at least $n - 2k + 2$ vertices.

The inspiration

Definition: k-connected

A graph is k-connected if and only if it has more than k vertices and does not have a vertex cut of size at most $k-1$.

Lemma

A graph is k-connected if and only if it has more than k vertices and for any subset U of $V(G)$, either $|N(U)| \geq k$ or $N[U] = V(G)$.

Mader, 1972

Every graph with average degree at least $4k$ has a $(k+1)$-connected subgraph with more than $2k$ vertices.

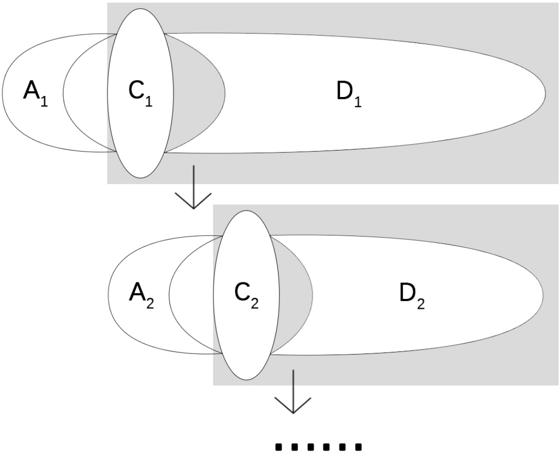
The Decomposition

Definition: $(f(k), k)$-decomposition

Let $k \in \mathbb{Z}^+$, $f(k)$ be a non-negative function on k. Let G be a graph on n vertices, where $n \geq f(k) + k$. We define an $(f(k), k)$-**decomposition** of G to be a sequence of triples $\{(A_i, C_i, D_i)\}$, $i \in [1, l]$, such that

1. $V(G)$ is a disjoint union of A_1, C_1, D_1
2. $C_i \cup D_i$ is a disjoint union of $A_{i+1}, C_{i+1}, D_{i+1}$, $i \in [1, l-1]$
3. $|C_i| \leq k - 1$, $i \in [1, l]$
4. $1 \leq |A_i| \leq |D_i|$, and there is no edge between A_i and D_i, $i \in [1, l]$
5. $|C_i| + |D_i| \geq n - f(k)$, $i \in [1, l-1]$
6. $|C_l| + |D_l| < n - f(k)$

The Decomposition



No k-connected subgraph $\Rightarrow$ Decomposition

Let G be a graph on n vertices. If G has no k-connected subgraph with at least $n - f(k)$ vertices, then G has a $(f(k), k)$-decomposition.

Definition: strong decomposition

Let $\{(A_i, C_i, D_i)\}$, $i \in [1, l]$ be an $(f(k), k)$-decomposition of G. We say the decomposition is strong if for any $i \in [1, l]$, $|A_i \cup C_i| < n - f(k)$.

Decomposition $\Rightarrow$ No k-connected subgraph

Let G be a graph on n vertices. If G has a strong $(f(k), k)$-decomposition, then G has no k-connected subgraph with at least $n - f(k)$ vertices.

The Proof

- R: red graph B: blue graph $R \cup B$ covers G
- Maximize R and B (Note: $R \cap B \neq \emptyset$)
- Suppose G has no monochromatic k-connected subgraph with at least $n - 2k + 2$ vertices.
- $\{A_i, C_i, D_i\}, i \in [1, l_R]$: decomposition in R
- $\{U_s, X_s, Y_s\}, s \in [1, l_B]$: decomposition in B
- $|A_i| \leq k - 1$, $|U_s| \leq k - 1$
- A_i, $[A_i, C_i]$, $C_{l_R} \cup D_{l_R} = A_{l_R+1}$ complete in red
- U_s, $[U_s, X_s]$, $X_{l_B} \cup Y_{l_B} = U_{l_B+1}$ complete in blue

The Proof

We use A to denote $\bigcup_{i=1}^{l_R} A_i$, and U to denote $\bigcup_{s=1}^{l_B} U_s$.

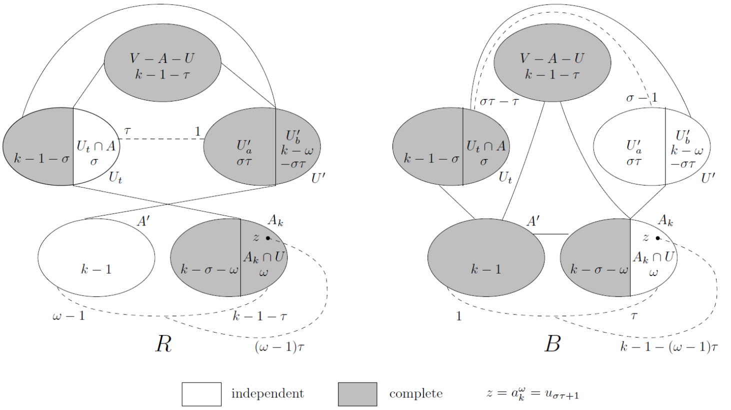
$$(k-1)(|A|+|U|)+\sum_{i=1}^{l_R+1}\binom{|A_i|}{2}+\sum_{s=1}^{l_B+1}\binom{|U_s|}{2}=|R|+|B|=\binom{n}{2}+|R\cap B|$$

The Proof

$$
\begin{aligned}
&(5k-3-n)|A\cap U|-(2k-1)-\frac{1}{2}|A\cap U|^2\\
=&(|A|-2k+1)(|U|-2k+1)+(|A|+|U|-4k+2)(k-|A\cap U|)\\
&+\sum_{i=1}^{l_R}\sum_{s=1}^{l_B}((k-1)|A_i\cap U_s|-\frac{1}{2}|A_i\cap U_s|^2-|Q(A_i\cap U_s)|)+|P|.
\end{aligned}
$$

- P consist of all edges that are both AC-type and UX-type, all the AC-type edges in $E(U_{l_B+1},U_{l_B+1})$, and all the UX-type edges in $E(A_{l_R+1},A_{l_R+1})$.
- Given a vertex $v\in A_i\cap U_s$, let $Q_R(v)$ be the family of edges uv with u in $A_i\cap Y_s$. Similarly, we define $Q_B(v)$. Let $Q(v)=Q_R(v)\cup Q_B(v)$.

The Counterexample



Future Works

Related Problems

- Matula's Problem
- Generalize to multicolored graphs
- Independence number and k-connected subgraph

Applications of the methods

- Vertex Partition
- Applications

The End